ELSEVIER

Theoretical Computer Science 180 (1997) 61–80

Theoretical Computer Science

Nonnegative companion matrices and star-height of $\mathbb{N}$-rational series

Frédérique Bassino *

I.G.M., Université de Marne-la-Vallée, 2, rue de la butte verte, 93166 Noisy le Grand Cedex, France

Received March 1996
Communicated by D. Perrin

Abstract

We prove a new result on $\mathbb{N}$-rational series in one variable. This result gives, under an appropriate hypothesis, a necessary and sufficient condition for an $\mathbb{N}$-rational series to be of star-height 1. The proof uses a theorem of Handelman on integral companion matrices.

1. Introduction

The notion of star-height was introduced by Eggan [4], and was later extended to rational formal power series in several noncommutative variables. In the case of formal languages, we know that the star-height is not bounded [4], but is decidable: Hashiguchi has shown the existence of an algorithm for determining it in this case (the proof is given in [7]). On the other hand, we have no similar result for rational formal power series. Reutenauer has studied in [13] the star-height in the tensor power series semiring.

The $\mathbb{N}$-rational series in one variable are particular formal power series, which we can define in several equivalent ways. One of these definitions gives the nth coefficient as the number of paths of length n in a graph G. Let $h(G)$ be the maximal number of loops necessarily fitted into each other in the graph G. The star-height is then equal to the minimum among the integers $h(G)$ when G runs over the set of graphs, which represent the same series. In this way, the series of star-height 0 are associated with acyclic graphs and are limited to polynomials. In this sense, the star-height can be interpreted as a measure of the loop complexity of series.

We know, from Soittola's theorem (see [12, 14]), that an $\mathbb{N}$-rational series in one variable is at most of star-height 2. This result has been obtained independently by Katayama et al. [8] by a different method.

* E-mail: bassino@univ-mlv.fr.

PII S0304-3975(96)00106-5

In this paper, we establish a property of series of star-height 1 which is sufficient for deciding the star-height of a particular class of series. Unlike Soittola's theorem, the condition that we give not only concerns the greatest real positive root of the series, but the set of all its real positive roots.

The star-height of the series in one variable is linked with various notions, in particular with that of the representation of the series by a matrix with polynomial entries. Each series of the class which particularly interests us in the following (the $\mathbb{N}$-rational series having a dominating root) can be represented by a matrix over $z\mathbb{N}[z]$ of size 2. We give also a characterization of the series of star-height 1 that uses their representation by matrices with polynomial entries (for a general overview of matrices with polynomial entries, one can refer to the survey written by Boyle [3]).

This paper is divided into three sections. In the first one, we present, by analogy with regular languages, some definitions and results about K-rational series in several noncommutative variables and we give an interpretation of the notion of star-height by making use of automata. The second section is devoted to Soittola's theorem and its consequences for $\mathbb{N}$-rational series in one variable, in particular for the star-height of these series. In the last section, using a result established by Handelman [6], we characterize spectral radii of irreducible matrices of size 1 having as only entry a polynomial of $z\mathbb{N}[z]$. They are strictly the algebraic integers greater than 1, of which some integral power satisfies the two following conditions: its modulus strictly exceeds the moduli of all of its algebraic conjugates and none of these conjugates is a real positive number. We call these algebraic integers Handelman numbers.

Next, we prove the main result: to have exclusively Handelman numbers as real positive roots is a necessary condition for an $\mathbb{N}$-rational series to be of star-height 1 and a sufficient one for deciding the star-height of an $\mathbb{N}$-rational series having moreover a dominating root. The proof by induction uses the linear representation of a rational series and the characterization of its real positive roots.

A preliminary version of this paper was presented at STACS 96 [1].

2. The analogies with regular languages

In this section, we draw the analogies between regular (or rational) languages and K-rational series.

2.1. Rational series

Let A be an alphabet and let K be a semiring. We call *rational expression* over A with multiplicity in K all the expressions written on $A \cup \{1\}$ with coefficients in K and using the three rational operators $+$,. and $*$. The *multiplicity* of a word in a rational expression is then the coefficient with which it appears in the expression.

As is the case of formal languages, a series is said to be *K-rational* if and only if it can be obtained from polynomials with coefficients in K, by making use of the three operations of sum, (convolution) product and the unary operation of *star*, defined by, when $P(0) = 0$,

$$P^{\star} = \frac{1}{(1-P)} = \sum_{n\geqslant 0} P^n.$$

With these conditions, rational languages are the rational series with coefficients in the boolean semiring.

A sequence is said to be K-rational if and only if its generating function is a K-rational series.

2.2. Recognizable series

We define an automaton $\mathfrak{A} = (Q,I,T,M)$ over A with multiplicity in K as follows. First of all, Q is the finite set of states of the automaton and I,T,M are matrices indexed by Q, such that $I \in K^{1\times n}$, $M \in K\langle A\rangle^{n\times n}$, $T \in K^{n\times 1}$, where n is the cardinality of Q and $K\langle A\rangle$ denotes the set of polynomials over A with coefficients in K. To be more precise, the entries of M are polynomials with degree at most 1. Moreover, if the (p,q)th entry of M is $M_{pq} = ka$ $(k \in K, a \in A)$, then there is an edge labelled by ka from the state p to the state q.

If c is the path

$$p \overset{k_1a_1}{\longrightarrow} q_1 \longrightarrow \cdots \longrightarrow q_{n-1} \overset{k_na_n}{\longrightarrow} q,$$

then its label is kw, where k is the product of the coefficients k_i and w is obtained by concatenation of the letters a_i, its length is the same one as that of the word w. Thus the behavior of the automaton is described by

$$\mathfrak{L}(\mathfrak{A}) = IM^*T \quad \text{with} \quad M^* = \sum_{i\geqslant 0} M^i.$$

Now, a formal series S belonging to $K\langle\langle A\rangle\rangle$ is said *recognizable* if it is the behavior of an automaton $\mathfrak{A} = (Q,I,T,M)$ over A with multiplicity in K, i.e.,

$$\mathfrak{L}(\mathfrak{A}) = S = IM^*T.$$

The triple (I,M,T) is then a *linear representation* of the series S.

2.3. The fundamental theorem

Kleene's theorem [9] showing the equivalence between rationality and recognizability for languages (the formal series with boolean coefficients) was extended by Schützenberger [15, 16] to formal series in several noncommutative variables with coefficients in an arbitrary semiring.

Theorem 1 (Schützenberger). *A formal series in several noncommutative variables is rational if and only if it is recognizable.*

2.4. Star-height

We define the *star-height* of a rational expression with multiplicity as the maximal number of nested stars in the expression and the *star-height* of a K-rational series as the minimum among the star-heights of the rational expressions (with multiplicity) describing the series.

More formally, we can give an algebraic definition, equivalent to the previous one, of the star-height of a K-rational series S. Let $(R_i)_{0 \leqslant i \leqslant n}$ be an increasing sequence of sets (with respect to inclusion) such that

- their union is the set of all rational series,
- the set of the polynomials is R_0,
- each R_i is stable under sum and product;
- furthermore, if $S \in R_i$ is proper, that is, $S(0) = 0$, then $S^\star \in R_{i+1}$.

The least integer n such that $S \in R_n$ is the star-height of the series S.

To interpret it by means of automata, we introduce the notion of cycle rank. We call *cycle* of an automaton any strongly connected component of the graph of states which is not reduced to a single element. We recursively define the *rank* of a cycle as follows:

A cycle is of rank 1 if all its loops go through a same state. It is of rank k, first, if there exists a state such that the graph obtained, by eliminating this state and all the edges incident to this state, contains a cycle of rank $k-1$ (all other cycles have to be of smaller rank) and second, if the choice of another state does not produce a graph having only cycles of rank strictly smaller than $k-1$.

The *cycle rank* of an automaton is then 0 if the graph of states is acyclic, otherwise it is equal to the maximum among the ranks of its cycles.

The following result gives an interpretation by means of automata of the star-height of a rational series.

Theorem 2. *The star-height of a rational series in several noncommutative variables is equal to the minimum among the cycle ranks of the automata (with multiplicity) recognizing the series.*

The proof of this theorem was given by Eggan [4, Theorem 4] for rational languages and still holds in the case of rational series with coefficients in an arbitrary semiring.

In that sense, the star-height can be interpreted as a measure of the loop complexity of the series.

Example 3. The Fibonacci sequence, of which the generating series is given by the rational expression of star-height 1

$$(x+x^2)^* = \frac{1}{1-(x+x^2)},$$

is recognized by the following automaton (Fig. 1) that has cycle rank 1. Therefore, the series is of star-height 1.

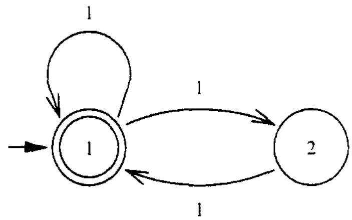

Fig. 1. Automaton recognizing the Fibonacci series.

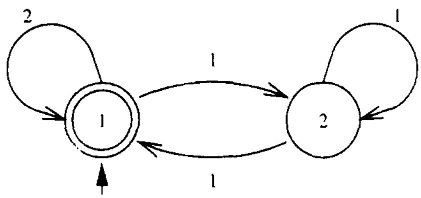

Fig. 2. Automaton recognizing the terms of even index in the Fibonacci series.

Example 4. The sequence of terms having an even index in the Fibonacci sequence can be recognized by an automaton (Fig. 2) of cycle rank 2 and described by the rational expression $(2x + x^2x^*)^*$, so it is at most of star-height 2. Indeed, it is exactly of star-height 2 (the justification of this assertion is given in Theorem 8).

3. About Soittola's theorem

A distinctive feature of $\mathbb{N}$-rational series is that they have an analytic characterization which is given by Soittola's theorem. The proof of this result allows us to also obtain a bound for the star-height.

3.1. Further definitions and results

To begin with, we mention some definitions and results about rational series (see [2, 14]). A sequence $r = (r_n)_{n\geqslant 0}$ of elements of a semiring K is called *K-rational* if there exists a triple (l, M, c) with $l \in K^{1\times n}$, $M \in K^{n\times n}$ and $c \in K^{n\times 1}$ (n being an integer greater than 1), such that

$$\forall k \geqslant 0 \quad r_k = lM^kc.$$

This triple is called a *linear representation* of the sequence r. One can note that the matrix M is the adjacency matrix of the graph of states of an automaton recognizing the series $\sum r_n z^n$.

When K *is a field*, a sequence is K-rational if and only if it satisfies a relation of linear recurrence:

$$r_n = \sum_{i=1}^{k} q_i r_{n-i} \quad (n \geqslant k).$$

The only monic polynomial

$$Q(z) = z^k - \sum_{i=1}^{k} q_i z^{k-i},$$

where k is minimal, is called the *minimal polynomial* of the sequence r over K and its roots are called the *roots* of r. If (l, M, c) is a reduced representation of r (i.e., M is of minimal size), the characteristic polynomial of the matrix (up to a multiplicative coefficient, since, by definition, the minimal polynomial of the series has to be monic).

A sequence r is equivalently K-rational if and only if its generating series

$$f_r(z) = \sum_{n \geqslant 0} r_n z^n$$

can be written as:

$$f_r(z) = \frac{P(z)}{Q(z)},$$

where $P, Q \in K[z]$ and $Q(0) = 1$. If P and Q are relatively prime, then the minimal polynomial of r is the *reciprocal polynomial* $\overline{Q}$ of Q,

$$\overline{Q}(z) = z^{degQ} Q(1/z).$$

When K is a field, this definition coincides with the previous one (Section 2.1) describing the rational series as an element of the rational closure of the polynomials with coefficients in K.

A polynomial Q has a *dominating root* if it has a real positive root α such that $\alpha > |\beta|$ for any other root β ($\beta \neq \alpha$) of Q. We say that a sequence r has a dominating root if its minimal polynomial does.

If K is a subfield of $\mathbb{C}$ and r a K-rational series which is not a polynomial, then for all sufficiently large n

$$r_n = \sum_{i=1}^{k} P_i(n) \alpha_i^n,$$

where the numbers α_i are the inverses of the distinct poles of the generating series $f_r(z)$ and each P_i is a polynomial whose coefficients are algebraic over K and whose degree is equal to one less than the multiplicity of the pole $1/\alpha_i$.

A sequence r is a *merge* of sequences $(s_i)_{0 \leqslant i \leqslant p-1}$ if, for $0 \leqslant i \leqslant p-1$ and for all positive integers n, $r_{pn+i} = s_{i,n}$. In terms of generating series:

$$f_r(z) = \sum_{i=0}^{p-1} z^i f_{s_i}(z^p).$$

Example 5. The sequence r, of which the generating series is

$$f_r(z) = \frac{1}{1 - z^2}$$

is a merge of the sequences of terms of even index and of terms of odd index in r given by their respective generating series

$$f_{s_0}(z) = \frac{1}{1-z} \quad \text{and} \quad f_{s_1}(z) = 0.$$

The notion of merge is natural, it allows us to consider only the case of series having a single pole on their convergence circle and, in this way, to avoid the asymptotic oscillatory behaviours of the coefficients of a series.

3.2. Soittola's theorem

The following result gives a characterization of K_+-rational series, the proof of which is given in [17,2,12]. The last one, that uses directly matrices instead of rational expressions, leads easily to an interpretation of this theorem by means of automata.

Theorem 6 (Soittola). *Let K be a subfield of $\mathbb{R}$. A sequence of positive numbers $r = (r_n)_{n \geqslant 0}$ is K_+-rational if and only if it is a merge of K-rational sequences having a dominating root.*

We recall that if K and L are two semirings and if $K \subset L$, L is then a *Fatou extension* of K if every L-rational series with coefficients in K is K-rational. From Fatou's lemma, an irreducible rational function

$$\frac{P(z)}{Q(z)} \quad \text{such that} \quad Q(0) = 1$$

and whose series expansion coefficients are integers, is the quotient of two polynomials with integral coefficients.

Therefore $\mathbb{Q}_+$ is a Fatou extension of $\mathbb{N}$, and a $\mathbb{Q}_+$-rational series with nonnegative integral coefficients is indeed $\mathbb{N}$-rational. Thus, a $\mathbb{Z}$-rational series with positive coefficients is $\mathbb{N}$-rational if and only if it is a merge of $\mathbb{Z}$-rational series having a dominating root.

Corollary 7. *It is decidable whether a $\mathbb{Z}$-rational series is $\mathbb{N}$-rational.*

The original proof of this result is given in [17, Corollary 5].

3.3. Consequence

A consequence of Soittola's theorem is that any $\mathbb{N}$-rational series in one variable is at most of star-height 2 [12]. This result has been independently established by Katayama et al. [8].

More precisely, it is possible to construct a linear representation of a $\mathbb{Z}$-rational series in one variable having nonnegative coefficients and a dominating root, in which any strongly connected component of the associated graph has the following shape (Fig. 3).

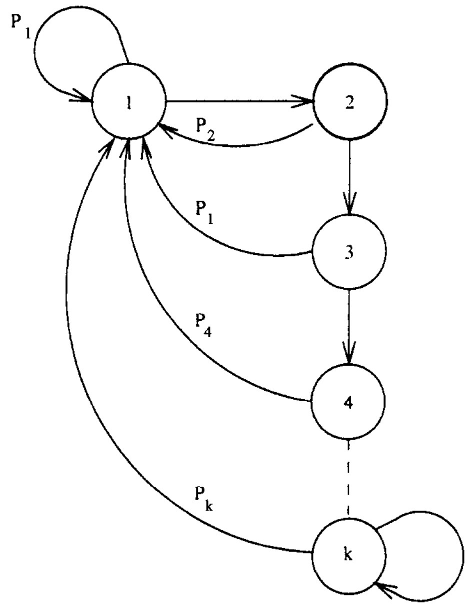

Fig. 3. Cycle of an automaton recognizing an ℕ-rational series having a dominating root.

Therefore the automaton accepting such a series is at most of cycle rank 2. As the star-height does not increase by merging series (the converse does not hold, see Remark 16), an ℕ-rational series is at most of star-height 2. Moreover, we show that this bound can be reached (Example 17).

4. Star-height 1

Knowing that the ℕ-rational series in one variable are at most of star-height 2 and that the polynomials are exactly those of star-height 0, we try to obtain a characterization of the series of star-height 1.

4.1. The main result

The following result gives a necessary condition for an ℕ-rational series to be of star-height 1 and allows us to decide its star height when it also has a dominating root.

Theorem 8. *Let r be an ℕ-rational series in one variable. If r is of star-height* 1, *then its real positive roots λ_i satisfy the following conditions*

$$\forall i, \quad \exists k_i \text{ such that} \begin{cases} \lambda_i^{k_i} \text{ is a Perron number} \\ \lambda_i^{k_i} \text{ has no positive real algebraic conjugate.} \end{cases}$$

If the series r also has a dominating root, these conditions are sufficient to characterize the series of star-height 1.

Henceforth, we call *Handelman numbers* the real positive numbers which satisfy the conditions of Theorem 8.

4.2. Linear algebra

In what follows, we shall exploit some properties of nonnegative matrices (see [5, Vol. 2] and [11]) to establish results from the linear representation of series.

4.2.1. Matrices with nonnegative entries

A square matrix is said to be *reducible* if, by a permutation, it can be reduced to a triangular block form; otherwise it is said to be *irreducible*.

According to the Perron–Frobenius theorem, such a matrix (irreducible) has a positive simple eigenvalue λ which is its spectral radius. If it has h eigenvalues of maximal modulus, the matrix is said to be *primitive* if $h = 1$, or is otherwise said to be *cyclic* of index h, and h is then called *the index of cyclicity* of the matrix. Moreover, a real number λ is the spectral radius of a primitive integral matrix if and only if it is a Perron number (a *Perron number* is a real algebraic integer exceeding 1 and strictly greater than the modulus of all of its algebraic conjugates). This property was established by Lind in [10].

The following result characterizes the spectral radius of a primitive integral companion matrix. For the proof one can refer to [6].

Theorem 9 (Handelman). *Let P be a monic polynomial of degree n with integral coefficients such that $P(0) \neq 0$. Then the polynomial P has only one real positive root λ, which is a simple root and strictly exceeds the moduli of all other roots of P if and only if, for all sufficiently large integers M and N, there exist positive integers T_i such that the companion matrix of the polynomial C belonging to $P\mathbb{Z}[z]$*

$$C = z^{N+M+n} - \sum_{i=0}^{N+n-1} T_i z^i$$

is primitive and has λ as its spectral radius.

In order to determine the star-height of an $\mathbb{N}$-rational series, we are only interested in the real positive roots of its minimal polynomial. In the proof of Theorem 8, we use the following lemma, which gives a characterization of the real positive roots of polynomials of the form: $z^n - \sum_{i=0}^{n-1} a_i z^i$ where the numbers a_i are nonnegative integers and $a_0 \neq 0$.

Lemma 10. *Let λ be a real positive number. Then λ is a root of a polynomial of the form: $z^n - \sum_{i=0}^{n-1} a_i z^i$, where the numbers a_i are nonnegative integers, and $a_0 \neq 0$ if and only if λ is a Handelman number.*

Proof. Let $P(z)$ be the polynomial $z^n - \sum_{i=0}^{n-1} a_i z^i$, where the numbers a_i are nonnegative integers and $a_0 \neq 0$. The polynomial P has only one real positive root λ which, therefore, has no real positive number as algebraic conjugate. Additionally, the conjugates λ_j of λ are roots of the polynomial P and thus have a modulus smaller than λ and, in case of equality, λ_j can be written as $\lambda\theta$ where θ is a root of unity. Let k be the common order of all these roots of unity, then λ^k is a Perron number and k is the index of cyclicity of the companion matrix of the polynomial P. Moreover, if

$$P(z) = z^n - \sum_{n_1 > \cdots > n_j} a_{n_i} z^{n_i} - a_0,$$

where $a_{n_1}, a_{n_2}, \ldots, a_{n_j}$ are the strictly positive numbers of the sequence $(a_i)_{1 \leqslant i}$, the index k is the greatest common divisor of the differences

$$n - n_1, n_1 - n_2, \ldots, n_k - 0$$

(the proof of this result, obtained by Frobenius, is given in [11, p. 34]. Thus all the exponents, that appear in the polynomial P are multiples of k, and we can make the change of variable $y = z^k$. We then obtain the polynomial $y^{n/k} - \sum_{i=0}^{n-1} a_i y^{i/k}$ and we deduce that λ^k has no real positive algebraic conjugate. Hence the only real positive root of the polynomial P is a Handelman number.

Conversely, let λ be a Handelman number. Let P be the minimal polynomial of λ^k; P is a monic polynomial of degree n with integral coefficients such that $P(0) \neq 0$ since $\lambda^k \neq 0$. Furthermore, the polynomial P has only one real positive root λ^k and λ^k is a simple root that strictly exceeds the moduli of the other roots of P. As λ^k is a Perron number, by Handelman's theorem, there exists a polynomial

$$C_0(z) = z^m - \sum_{i=0}^{m-1} t_i z^i,$$

where the numbers t_i are nonnegative integers, such that the companion matrix of the polynomial C_0 is primitive. Thus, the polynomial C defined by

$$C(z) = C_0(z^k)$$

is the polynomial that we look for, concluding the proof of the lemma. □

Corollary 11. *Let M be a nonzero and irreducible integral matrix, whose spectral radius λ is a Handelman number, and let P be its characteristic polynomial. Then, in the ideal $P\mathbb{Z}[z]$, there exists a polynomial*

$$C(z) = z^n - \sum T_i z^i, \quad T_i \in \mathbb{N}, \quad C(0) \neq 0$$

whose companion matrix is irreducible and has λ as spectral radius.

Proof. Let P be the characteristic polynomial of the matrix M. Let k be the index of cyclicity of the matrix M, k is the greatest common divisor of the differences between two exponents that successively appear in the polynomial P, thus all these exponents are multiples of k and we can make the change of variable $y = z^k$ in the polynomial P. The polynomial obtained in this way satisfies the conditions of Handelman's theorem the application of which leads to the announced result. The reciprocal change of variable is sufficient to obtain the matrix that we need. □

The Handelman numbers are exactly the numbers that can be spectral radii of irreducible integral companion matrices (Fig. 4).

Therefore we obtain automata of cycle rank 1; moreover, as the matrices are irreducible, the graphs of states of the automata are strongly connected.

4.2.2. Matrices with polynomial entries

For a more general presentation of this subject, one can refer to [3].

We know that an $n \times n$-matrix over $\mathbb{N}$ can be considered as the adjacency matrix of a directed graph with n vertices.

However there is a still more general way to present a directed graph by means of matrices with entries in $z\mathbb{N}[z]$ (the polynomials in one variable without a nonzero constant term). This allows us to obtain a more concise presentation and to use additional algebraic arguments.

Let M be a matrix of size n over $z\mathbb{N}[z]$. We construct a directed graph associated with M as follows. The set of vertices of the graph contains a subset of n vertices (say $1, 2, \ldots, n$), which index the rows and the columns of M. Moreover, if the (i,j)-th entry of M is $M_{ij} = \sum_k a_k z^k$, we construct a_k paths of length k from the vertex i to the vertex j, in such a way that every interior vertex of these paths (a path of length n has $k-1$ interior vertices) has just one incoming edge and one outgoing edge and

companion matrix

$$\begin{pmatrix} P_1 & 1\;0 \ldots 0 \\ \vdots & 0\;1 \ddots \vdots \\ \vdots & \vdots\;0 \ddots 0 \\ P_{k-1} & \vdots\;\vdots \ddots 1 \\ P_k & 0\;0 \ldots 0 \end{pmatrix}$$

Matrix with polynomial entries

$$\left(\sum_{i=1}^{k} P_i z^i\right)$$

Flower automaton associated with a Handelman number

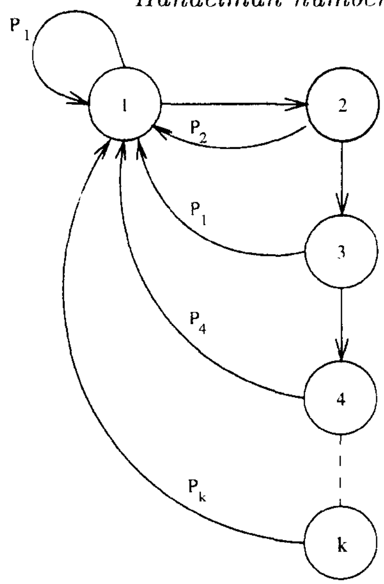

Fig. 4. Representations associated with a Handelman number.

is disjoint from the set of the distinguished vertices $1, 2, \dots, n$. This process produces a graph which can have much more than n vertices (hence the conciseness of the presentation). The associated matrix is then a $(0,1)$-matrix. Another way to represent $a_k z^k$ consists in constructing one path only with the same condition on the interior vertices and with all but one edge labelled by 1 and the last one by a_k, the associated matrix has then nonnegative integral entries.

Example 12. The matrix

$$M = \begin{pmatrix} 0 & z^2 \\ 2z^3 & z^3 \end{pmatrix}$$

gives a directed graph (Fig. 5).

The n distinguished vertices (corresponding to the indices of the rows and of the columns of M) in the graph form a "rome", i.e., any sufficiently long path in the graph goes through at least one of the vertices of the rome.

Conversely, given a rome in a directed graph, one can reverse the procedure and obtain a matrix presentation M with entries in $z\mathbb{N}[z]$, where M has size n, if n is the cardinality of the rome.

Remark 13. The rank of a cycle (i.e., a strongly connected component of the graph associated with an automaton, see Section 2.4) is equal to the cardinality of the smallest romes, producing a polynomial presentation of this strongly connected component. Thus, the cycle rank of an automaton is the maximum among the cardinalities of the smallest romes of its cycles.

Another point of view consists in considering the matrix M as giving a directed graph G, whose edges are labelled by polynomials without a nonzero constant term. The number of edges from a vertex i to a vertex j is then the (i,j)-th entry of M evaluated at $t = 1$. Each power of z and each corresponding coefficient a_k, that appear in the label of an edge, give the length and the number of the associated paths.

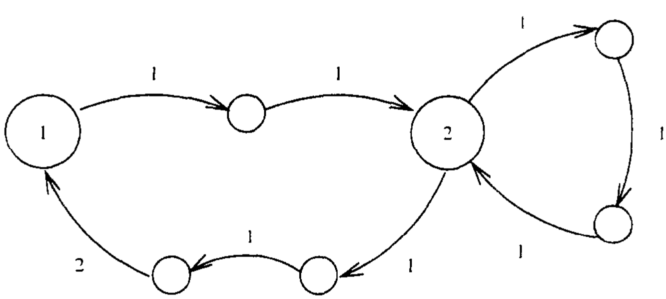

Fig. 5.

If N is a matrix over $\mathbb{N}$, which is the adjacency matrix of a directed graph, then zN is a matrix over $z\mathbb{N}[z]$ which presents the graph in this more general way.

Proposition 14. *If M and N are matrices over $z\mathbb{N}[z]$ and over $\mathbb{N}$, respectively, which present the same directed graph G, then*

$$\det(I - zN) = \det(I - M).$$

Now we can prove Theorem 8. To begin with, we give several equivalent characterizations of the series of star-height 1, from which we deduce that the condition of Theorem 8 is necessary.

4.3. Proof of the theorem (necessity)

Proposition 15. *Let $r = \sum_{n\geqslant 0} r_n z^n$ be an $\mathbb{N}$-rational series, then r has star-height 1 if and only if one the three following conditions is satisfied*

(i) *the series r can be written as*

$$\sum_{i\in I} \frac{P_i}{\prod_{j\in J_i}(1 - Q_{i_j})} \qquad \forall i,j \quad P_i \in \mathbb{N}[z], \quad Q_{i_j} \in z\mathbb{N}[z], \tag{1}$$

where I,J are finite sets and at least one of the polynomials Q_{i_j} is not zero.

(ii) *the series r admits a linear representation (l,M,c) with, n being an integer greater than 1, $l \in \mathbb{N}^{1\times n}$, $M \in \mathbb{N}^{n\times n}$ and $c \in \mathbb{N}^{n\times 1}$, such that*
- $\forall k \geqslant 0$ $r_k = lM^k c$,
- *The matrix M has a block triangular form and its diagonal blocks are irreducible companion matrices.*

(iii) *the series r admits a polynomial representation (l,M,c) with, $n\geqslant 1$, $l \in \mathbb{N}[z]^{1\times n}$, $M \in z\mathbb{N}[z]^{n\times n}$ and $c \in \mathbb{N}[z]^{n\times 1}$, such that*
- $r = \sum_{k=0}^{\infty} lM^k c$,
- *The matrix M is triangular.*

Proof. The series defined by one of the three previous conditions have star-height one. In the first case, they are given by a rational expression of star-height 1:

$$\sum_{i\in I} P_i \prod_{j\in J} Q_{i_j}^{*}.$$

Otherwise, the automaton associated with the matrix M has cycle rank 1, since each of its strongly connected components has rank 1.

It remains to be proved that the converse holds.

r has star-height 1 $\Rightarrow$ i: Let R_0 be the set of polynomials with coefficients in $\mathbb{N}$, and R'_1 the set of $\mathbb{N}$-rational series of star-height 1. The set of series of form (1) is included in R'_1, and contains each element $P^{\star}$ where $P \in R_0$ and $P(0) = 0$. Additionally, this

set is stable for the sum and the (convolution) product, and we thus conclude that the expression (1) characterizes the $\mathbb{N}$-rational series of star-height 1.

i $\Rightarrow$ iii: If

$$r_i = \frac{P_i}{\prod_{j\in J_i}(1-Q_{i_j})},$$

the series $r_i z^{n_i}$ where $n_i = \mathrm{Card}(J_i)$ admits the polynomial representation (l_i, M_i, c_i) with

$$l_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ P_i \end{pmatrix}^{t}, \quad M_i = \begin{pmatrix} Q_{i1} & 0 & \dots & \dots & 0 \\ z & Q_{i2} & \ddots & & \vdots \\ 0 & z & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & 0 & z & Q_{in_i} \end{pmatrix} \quad \text{and} \quad c_i = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Therefore, if $r = \sum_{i=1}^{N} r_i$, the series rz^n, where $n = \max_{1\leqslant i\leqslant N}\{n_i\}$, admits then the polynomial representation (l, M, c) with

$$l = \begin{pmatrix} l_1 z^{n-n_1} \\ l_2 z^{n-n_2} \\ \vdots \\ l_N z^{n-n_N} \end{pmatrix}^{t}, \quad M = \begin{pmatrix} M_1 & 0 & \dots & \dots & 0 \\ 0 & M_2 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 0 & M_N \end{pmatrix} \quad \text{and} \quad c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{pmatrix}.$$

Up to a shift of indices, any series given by the formula (1) has a polynomial representation satisfying (iii).

iii $\Rightarrow$ ii: In order to obtain a linear representation (l, M, c) from a polynomial one, say (l_P, M_P, c_P), we construct from the matrix M_P a directed graph in the following way. The set of vertices contains a subset of n vertices (say $1, 2, \dots, n$) which index the rows and the columns of M_P. If the (i, j)-th entry of the matrix M_P is $M_{ij} = \sum_k a_k z^k$, we construct, for any k, a path of length k from the vertex i to the vertex j, in such a way that the set of all interior vertices of this path is disjoint from the set of the distinguished vertices $1, 2, \dots, n$ and that each of these new vertices has just one incoming edge and one outgoing edge, all but one labelled by 1, the remaining one having a_k as label. The entries of l_P and of c_P are expanded in the same way, moreover the vertices where begin (resp. end) the paths corresponding to the polynomial entries of l_P (resp c_P), become the initial (resp. final) states (the corresponding entries are 1) of the automaton. The linear representation, that we look for, is obtained directly from this automaton, concluding the proof. $\square$

We deduce, from the Proposition 15, that the real positive roots of a series of star-height 1 are, each, a root of one of the reciprocal polynomials $\overline{(1-Q_{i_j})}$ or, equivalently, a spectral radius of an irreducible integral companion matrix. Thus, by Lemma 10, they are algebraic integers, of which an integral power is a Perron number and has no real positive algebraic conjugate, and are hence Handelman numbers.

Remark 16. We can not use the characterization of $\mathbb{N}$-rational series by a merge of $\mathbb{Z}$-rational series having a dominating root for the converse proof. Actually, the star-height of series that appear in the merge do not give information on the star-height of the initial one. This leads us to suppose the existence of a dominating root, an assumption that we use to prove that the auxiliary series introduced in the proof are $\mathbb{N}$-rational.

Example 17. The series defined by the function

$$f_r(z) = \frac{1}{(1-9z^2)(1-z-z^2)}$$

is of star-height 1, and the square matrix of its minimal linear representation is cyclic of index 2. On the other hand, the series of terms of even index given by the function

$$h(z) = \frac{1-z}{(1-9z)(1-3z+z^2)}$$

is of star-height 2, because it has two real positive conjugated roots

$$\frac{(3+\sqrt{5})}{2} \quad \text{and} \quad \frac{(3-\sqrt{5})}{2}.$$

4.4. *Proof of the theorem (sufficiency)*

Conversely, let r be an $\mathbb{N}$-rational series having a dominating root. We suppose that its real positive roots are Handelman numbers, that is that they satisfy the following conditions

$$\forall i, \quad \exists k_i \text{ such that} \begin{cases} \lambda_i^{k_i} \text{ is a Perron number} \\ \lambda_i^{k_i} \text{ has no positive real algebraic conjugate.} \end{cases}$$

The aim is to establish that such a series is of star-height 1.

Method: The proof is made by induction on the number of real positive roots of the series. We construct, cycle by cycle, an automaton in such a way that each of its strongly connected components has rank 1. We essentially use the linear representation of series and the characterization of spectral radii of irreducible integral companion matrices, which produce automata of cycle rank 1.

As r is an $\mathbb{N}$-rational series, it has a representation (l, M, c), where

$$l \in \mathbb{N}^{1\times p}, \qquad M \in \mathbb{N}^{p\times p} \quad \text{and} \quad c \in \mathbb{N}^{p\times 1},$$

such that:

$$\forall k \geqslant 0 \quad r_k = lM^k c.$$

Up to a possible permutation, we may suppose that the matrix M is of the form

$$\begin{pmatrix} M_{11} & 0 & \dots & 0 \\ M_{21} & M_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ M_{m1} & \dots & M_{mm-1} & M_{mm} \end{pmatrix}$$

where the square matrices M_{ii} are nonzero and irreducible. Then the characteristic polynomial of the matrix M can be written as the product of M_i, the characteristic polynomials of the matrices M_{ii}. From Corollary 11, we can construct an irreducible integral companion matrix having the same spectral radius as M_{ii} and whose characteristic polynomial C_i belongs to the ideal $M_i\mathbb{Z}[z]$. If the matrix M_{ii} is primitive, the matrix constructed is also primitive.

In these conditions, the generating series $f_r(z)$ of the series r can be written as:

$$f_r(z) = \frac{P(z)}{\prod_{i=1}^{m}(1 - Q_i)(z)},$$

where $P \in \mathbb{Z}[z]$ and $\forall i$, $1 - Q_i$ is the reciprocal polynomial of C_i and thus $Q_i \in \mathbb{N}[z]$.

We reason by induction on the number m of real positive roots of the series r. Let $\lambda_1, \lambda_2, \dots, \lambda_m$ be these roots arranged in increasing order (a multiple root occurs in this sequence as many time as its multiplicity). We set

$$(1 - Q_1)(z) = 1 - \sum_{j=1}^{q} q_j z^j,$$

and we define the sequence $(s_n)_{n \geqslant q}$ as:

$$s_n = r_n - \sum_{j=1}^{q} q_j r_{n-j} \quad (\forall n \geqslant q).$$

If $m = 1$, then $s_n = 0$ for $n \geqslant h = \max(\deg P + 1, q)$ and

$$r_{n+h} = lM^n c \quad \forall n \geqslant 0,$$

where

$$l = \begin{pmatrix} r_{h+q-1} \\ \vdots \\ r_{h+1} \\ r_h \end{pmatrix}^{\mathrm{t}}, \qquad M = \begin{pmatrix} q_1 & 1 & 0 & \dots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ q_{q-1} & \vdots & \vdots & \ddots & 1 \\ q_q & 0 & 0 & \dots & 0 \end{pmatrix} \quad \text{and} \quad c = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

We then obtain the following equality

$$\sum_{n \geqslant h} r_n z^n = \frac{1}{1 - Q}\Big(P - (1 - Q) \sum_{n=0}^{h-1} r_n z^n\Big),$$

where $1-Q$ is the reciprocal polynomial of the characteristic polynomial of the matrix M and thus $Q \in \mathbb{N}[z]$. The expression $(P-(1-Q)\sum_{n=0}^{h-1} r_n z^n)$ is of degree greater than h, so the only remaining terms result from the product $Q\sum_{n=0}^{h-1} r_n z^n$, and the numerator of the fraction is a polynomial with nonnegative coefficients. Hence, the series r is of star height 1.

We now suppose that the result is true for an $\mathbb{N}$-rational series having a dominating root and exactly $(m-1)$ Handelman numbers as real positive roots.

We shall show that the series s verifies the hypothesis of induction stated above, and we shall then be able to prove the announced result.

First, we must show that there exists a nonnegative integer h such that

$$\forall n \geqslant h \quad s_n \geqslant 0.$$

Since the series r is $\mathbb{N}$-rational, for all sufficiently large n

$$r_n = \sum_{0 \leqslant i \leqslant p} A_i(n)\alpha_i^n \quad \forall n \geqslant n_0,$$

where the numbers α_i are the inverses of the distinct poles of the generating function $f_r(z)$ of r, and each A_i is a polynomial whose degree is equal to one less than the multiplicity of the pole $1/\alpha_i$.

We then use the formula:

$$(*) \qquad n^j\alpha_i^n - q_1(n-1)^j\alpha_i^{n-1} - \cdots - q_q(n-q)^j\alpha_i^{n-q}$$

$$= n^j\alpha_i^n\left(1-\sum_{k=1}^{q}\frac{q_k}{\alpha_i^k}\right) + jn^{j-1}\alpha_i^n\sum_{k=1}^{q}\frac{kq_k}{\alpha_i^k} + B(n)\alpha_i^n,$$

where B is a polynomial of degree $j-2$.

Since $(r_n)_{n\geqslant n_0}$ is a linear combination of series of the form

$$\sum_{n\geqslant n_0} n^j\alpha_i^n z^n,$$

the series $\sum_{n\geqslant n_0+q} s_n z^n$ is a linear combination of series of the form

$$\sum_{n\geqslant n_0+q} [n^j\alpha_i^n - q_1(n-1)^j\alpha_i^{n-1} - \cdots - q_q(n-q)^j\alpha_i^{n-q}]z^n.$$

We deduce that

$$\forall n \geqslant n_0+q, \qquad s_n = \sum_{0\leqslant i\leqslant p} B_i(n)\alpha_i^n.$$

We set $\alpha_0 = \lambda_m$. Since λ_m is the dominating root of the series r,

$$\forall i \in \{1,2,\ldots,p\}, \quad \lambda_m > |\alpha_i|.$$

Let β be the order of multiplicity of λ_m in the polynomial $\overline{1-Q}$ and let a_0 be the leading coefficient of the polynomial A_0,

$$r_n \sim_{n\to\infty} a_0 n^{\beta-1}\lambda_m^n.$$

since $r_n \geqslant 0$ and r is not a polynomial, $a_0 > 0$. We can now compute the equivalent of s_n when n tends toward infinity.

If $\lambda_1 \neq \lambda_m$, the leading coefficient of B_0 is, from $(*)$,

$$a_0 \lambda_m^n \left(1 - \sum_{i=1}^{q} \frac{q_i}{\lambda_m^i}\right) = a_0 \lambda_m^n (1 - Q_1)(1/\lambda_m).$$

Furthermore, $(1 - Q_1)(1/\lambda_m) > 0$ because this polynomial has only one real positive root, $1/\lambda_1$, and $1/\lambda_1 > 1/\lambda_m$. Since

$$s_n \sim_{n \to \infty} a_0 \lambda_m^n (1 - Q_1)(1/\lambda_m) n^{\beta - 1},$$

we can deduce that for all sufficiently large n, $n \geqslant h$, $s_n \geqslant 0$.

If $\lambda_1 = \lambda_m$, $\beta = m$, since $1/\lambda_1$ is a root of the polynomial $(1 - Q_1)$, the polynomial B_0 is of degree $m - 2$ and its leading coefficient then is

$$a_0 (m - 1) \lambda_m^n \sum_{i=1}^{q} \frac{i q_i}{\lambda_m^i} > 0,$$

by $(*)$. Since

$$s_n \sim_{n \to \infty} a_0 (m - 1) \lambda_m^n \sum_{i=1}^{q} \frac{i q_i}{\lambda_m^i} n^{m-2},$$

we can deduce that for all sufficiently large n, $n \geqslant h$, $s_n \geqslant 0$.

We have shown that the sequence $(s_n)_{n \geqslant h}$ has nonnegative coefficients. Additionally, the generating function of the series s can be written as

$$f_s(z) = f_r(z)(1 - Q_1) - W(z),$$

where W is a polynomial of degree $(q - 1)$, so the series s is $\mathbb{Z}$-rational and has λ_m as dominating root. Since the series s has $(m - 1)$ Handelman numbers as only real positive roots, by the hypothesis of induction, the series s is $\mathbb{N}$-rational of star-height 1. So, there exists a linear representation (l_s, M_s, c_s) in which the entries of matrices l_s, M_s and c_s are nonnegative integers, such that

$$s_{n+h} = l_s {M_s}^n c_s \quad \forall n \geqslant 0$$

and matrix M_s is associated with an automaton of cycle rank 1 (that is to say, we can obtain from this automaton an acyclic automaton by eliminating a state and all edges incident to this state).

Finally, we consider the triple (L, N, C) consisting of

$$L = \begin{pmatrix} l_s \\ \hline r_{h+q} \\ \vdots \\ r_{h+1} \\ r_h \end{pmatrix}^{t}, \qquad N = \left(\begin{array}{c|cccccc} M_s & c_s & 0 & \dots & \dots & 0 \\ \hline 0 & q_1 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & 0 & \ddots & 0 \\ \vdots & q_{q-1} & \vdots & \vdots & \ddots & 1 \\ 0 & q_q & 0 & 0 & \dots & 0 \end{array}\right) \quad \text{and} \quad C = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

We verify by induction on n that, for all $n \geqslant 0$,

$$LN^n = (l_s M_s^n \mid r_{n+h+q}, \ldots, r_{n+q})$$

and we deduce that

$$r_{n+h} = LN^n C \quad \forall n \geqslant 0$$

concluding the proof that the series r is of star-height 1.

4.5. Decidability

The decidability of the star-height in the case of $\mathbb{N}$-rational series having a dominating root comes from the possibility of calculating, for each positive real root λ, the maximal integer k such that λ^k satisfies the conditions of Theorem 8. The method is similar to that used to decide the $\mathbb{N}$-rationality of a $\mathbb{Z}$-rational series (see [14, p. 74–75]). We construct the polynomial with integral coefficients

$$T(x) = \prod_{i,j} (\beta_i x - \beta_j),$$

where β_i runs over the set of roots having λ as modulus. Next, we determine the integer n_T such that

$$\forall n > n_T, \quad \phi(n) > \deg(T)$$

where ϕ is the Euler's function and we factor the polynomial T as a product of cyclotomic polynomials of order smaller than n_T. The integer k is then the l.c.m. of the orders of the cyclotomic polynomials which appear in the factorization of T.

Remark 18. The previous Theorem 8 allows us to decide the star-height of some $\mathbb{N}$-rational series if no new positive root appears in their writing as a merge of $\mathbb{Z}$-rational series. This happens, for example, when a series has a multiple periodic but unique root ($f_r(z) = 1/(1-z^n)^q$).

We may conjecture that to have nothing but Handelman numbers as real positive roots characterizes $\mathbb{N}$-rational series of star-height 1.

Acknowledgements

I wish to thank Dominique Perrin for many helpful discussions that greatly improved this paper. Thanks also to Mike Boyle for having called my attention to matrices with polynomial entries.

References

[1] F. Bassino, Star-height of an ℕ-rational series, in: C. Puech and R. Reischuk, eds., *STACS 96*, No. 1046, Lecture Note in Computer Science, Vol. 1046 (Springer, Berlin, 1996) 125–135.

[2] J. Berstel and C. Reutenauer, *Rational Series and Their Languages* (Springer, Berlin, 1988).

[3] M. Boyle, Symbolic dynamics and matrices, in: R. Brualdi, S. Friedland and V. Klee, eds., *Combinatorial and Graph-Theoretic Problems in Linear Algebra*, IMA Volumes in Mathematics and Its Applications, Vol. 50 (Springer, Berlin, 1994).

[4] L.C. Eggan, Transition graphs and star height of regular events, *Michigan Math. J.* **10** (1963) 385–397.

[5] F.R. Gantmacher, *The Theory of Matrices* (Chelsea, New York, 1959).

[6] D.E. Handelman, Spectral radii of primitive integral companion matrices and log-concave polynomials, in: Peter Walters, ed., *Symbolic Dynamic and its Applications*, Contempory Mathematics, Vol. 135 (1992) 231–238.

[7] K. Hashiguchi, Relative star-height, star-height and finite automata with distance functions, in: J.-E. Pin, ed., *Formal Properties of Finite Automata and Applications*, Lecture Notes in Computer Sciences, Vol. 386 (Springer, Berlin, 1989) 74–88.

[8] T. Katayama, M. Okamoto and H. Enomoto, Characterization of the structure-generating functions of regular sets and DOL growth functions, *Inform. and Control* **36** (1978) 85–101.

[9] S.C. Kleene, Representation of events in nerve nets and finite automata, in: C. Shannon and J. McCarthy, eds., *Automata Studies* (Princeton Univ. Press, Princeton, NJ, 1956) 3–41.

[10] D. Lind, The entropies of topological Markov shifts and a related class of algebraic integers, *Ergod. Theory Dynam. Syst.* **4** (1984) 283–300.

[11] H. Minc, *Nonnegative Matrices* (Wiley, New York, 1988).

[12] D. Perrin, On positive matrices, *Theoret. Comput. Sci.* **94** (1992) 357–366.

[13] C. Reutenauer, Inversion height in free fields, *Selecta Mathematica. New Series* 2(1) (1996) 1–18.

[14] A. Salomaa and M. Soittola, *Automata Theoretic Aspect of Formal Power Series* (Springer, Berlin, 1978).

[15] M.P. Schützenberger, On the definition of a family of automata, *Inform. and Control* **4** (1961) 245–270.

[16] M.P. Schützenberger, Certain elementary families of automata, in: *Symp. on Math. theory of automata*, Polytechnic Institute of Brooklyn (1962) 139–153.

[17] M. Soittola, Positive rational sequences, *Theoret. Comput. Sci.* **2** (1976) 317–322.